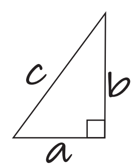
What to bring to class:
Ask students to bring PM 4A and 5A.

9.4 Newtons Method and $\sqrt{2}$

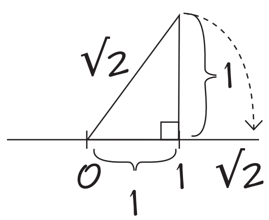
[SAY: We've learned: not all numbers are rational! So far, looks like irrational #'s are just theoretical since we can't even write one down (infinitely many digits)
In fact, any building contractor will tell you the opposite – irrational #'s occur naturally and are used frequently.]

<u>Square roots</u> – are often irrational. [SAY: To understand, we'll look at $\sqrt{2}$ in detail.]

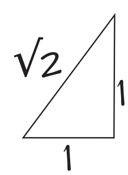
1. <u>Finding $\sqrt{2}$</u>: Pythagorean Thm:



$a^2 + b^2 = c^2$



$\Longrightarrow$



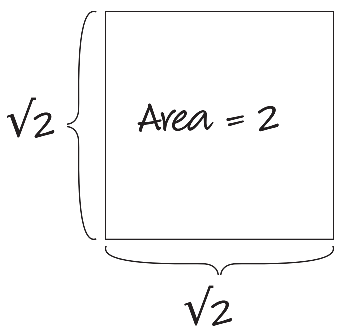
$1^2 + 1^2 = (\sqrt{2})^2$

$\sqrt{2}$ is a pt on the # line $\Longrightarrow$ its a real number.

II. Find an infinite decimal expansion

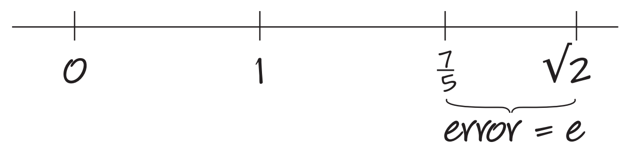
Construct a square of Area = 2

[ASK: How long is each side?]



Approximate $\sqrt{2} \approx \frac{7}{5} = 1.4$

$(1.4^2 = 1.96 < 2)$



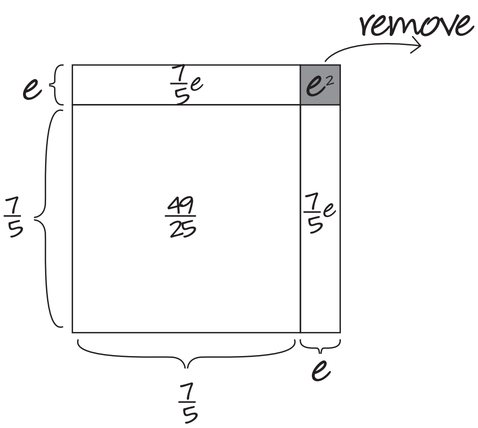
$\sqrt{2} = \frac{7}{5} + e$

$2 = (\frac{7}{5} + e)^2 = \frac{49}{25} + 2 \cdot \frac{7}{5} \cdot e + e^2$ (remove e^2)

$2 \approx \frac{49}{25} + \frac{14}{5} e$ (×25)

Solve for e $\Longrightarrow 50 \approx 49 + 70e$

$\boxed{e \approx \frac{1}{70}}$



Hence a better estimate for $\sqrt{2}$

$\sqrt{2} = \frac{7}{5} + e \approx \frac{7}{5} + \frac{1}{70} = \frac{99}{70} = 1.414$

← A better estimate!

Algebraically: Approximate by a (instead of $\frac{7}{5}$)

$\sqrt{2} = a + e \Longrightarrow 2 = a^2 + 2ae + e^2$ (remove e^2)

solve for e $\Longrightarrow 2 - a^2 \approx 2ae \Longrightarrow \frac{2 - a^2}{2a} \approx e$

$\Longrightarrow e \approx \frac{1}{a} - \frac{a}{2}$

New Approximation:

$a_{new} = a + e = a + \frac{1}{a} - \frac{a}{2} = \frac{a}{2} + \frac{1}{a}$

Applying repeatedly gives:

<u>Approx</u>:

$1 \longrightarrow \frac{1}{2} + 1 = \frac{3}{2} = 1.5$

$1.5 \longrightarrow \frac{3}{4} + \frac{2}{3} = \frac{17}{12} = 1.41\overline{6}$

$\frac{17}{12} \longrightarrow \frac{17}{24} + \frac{12}{17} = \frac{577}{408} = 1.41\overline{42156862745098039}$

(open book — pointing to the repeating digits)

better and better approx.

Note: ① $\sqrt{2}$ is <u>not</u> 1.414213562 (calc read) followed by ∞ # of random digits.

(as many students believe)

② In each step there is always error > 0. Hence it appears that $\sqrt{2}$ is not rational. (if $e = 0$, then we would get $\sqrt{2}$ rat.)

<u>Fact</u>: $n = P_1 P_2 P_3 \ldots P_k$ is a prime factorization. Then $n^2 = (P_1 \ldots P_k)(P_1 \ldots P_k)$ has an even # (2k's worth) of primes in its P.F.

<u>Thm</u> $\sqrt{2}$ is irrational.

<u>Proof</u>: Suppose $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{a}{b} \text{ for some } a, b \text{ whole \#'s.}$$

Square both sides: $2 = \frac{a^2}{b^2}$

$$\Longrightarrow \underbrace{a^2}_{\text{even \# of primes in P.F.}} = \underbrace{2b^2}_{\text{even \# +1 number of primes in P.F.}}$$

even # +1 number of primes in P.F. → odd # of primes!

The same number can't have both an even number and odd # of primes in its P.F.

Contradictions. $\sqrt{2}$ is <u>not</u> rational $\Longrightarrow$ $\sqrt{2}$ is irrational.

Thm If a whole # n is not a square ($n \neq 1, 4, 9, 16, \ldots$)

then $\sqrt{n}$ is irrational

HW Read § 9.4 Do HW set 40

Radical Rules:

(1) $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$

(2) $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$

Proof of 2: $\sqrt[n]{a} = x,\ \sqrt[n]{b} = y \iff x^n = a,\ y^n = b$

Then $ab = x^n y^n = (xy)^n$

PR4

By def: $\sqrt[n]{ab} = xy = \sqrt[n]{a}\,\sqrt[n]{b}$ ■

[SAY: Radical Rule 2 follow from PR4 - Can we write $\sqrt[n]{a}$ as an exponent?]

Suppose we could write $\sqrt{3} = 3^{\square}$ some exponent

what is $\square$?

We know $(\sqrt{3})^2 = 3^1 \Longrightarrow 3^1 = (3^{\square})^2 = 3^{2\cdot\square}$

Equating we get $2 \cdot \square = 1 \Longrightarrow \square = \frac{1}{2}$!

$\sqrt{3} = 3^{\frac{1}{2}} \longleftarrow$ fraction exponents!

Def: Let a be any non negative real number and n a positive integer; then

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

Rule 1

$$\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt{a^n})^m$$

Rule 2

$$\sqrt[n]{a}\,\sqrt[n]{b} = a^{\frac{1}{n}} b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} = \sqrt[n]{ab}$$

Ex

$$36^{3/2} = (\sqrt{36})^3 = 6^3 = 216$$

$$1024^{6/20} = 1024^{3/10} = (\sqrt[10]{1024})^3 = 2^3 = 8$$

HW Read § 9.4 Do HW set 39

(Don't do 1 or 9).