What to bring to class:
Ask students to bring PM 4A and 5A.

9.3 Rational and Real #'s [$1\frac{1}{2}$ days]

(1) Divisions like $5 \div 3$ did not have whole # solutions

$\Longrightarrow$ Enlarged whole #'s to get fractions:

$$5 \div 3 = \frac{5}{3}$$

New Property: Multiplicative Inverse: $x \cdot \frac{1}{x} = 1$

(2) Subtraction like $2 - 8$ did not have a solution in whole #'s or fractions.

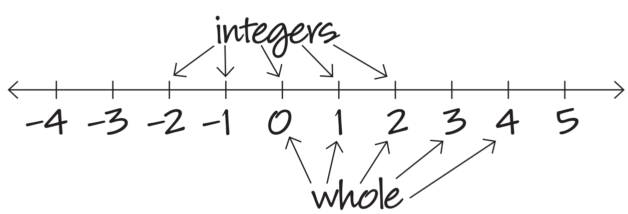
$\Longrightarrow$ Enlarge whole numbers to get integers.

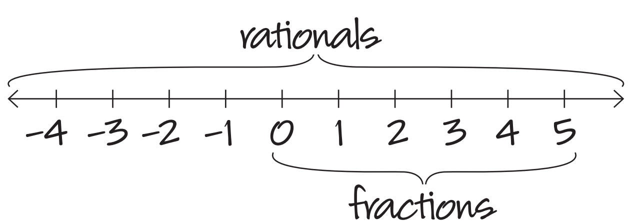
New Property: Additive inverse: $a + -a = 0$.

Doing both gives

<u>Def</u>: The <u>rationals</u> are the set of fractions together with their opposites.

<u>Ex</u> $\frac{-3}{8}$ is a rational, but not a fraction or integer.





Rationals satisfy the <u>complete list of Arithmetic Prop.</u>

- Commutative Property $a + b = b + a,\ ab = ba$
- Associative Property $a + (b + c) = (a + b) + c$ $a(bc) = (ab)c$
- Distributive Property $a(b + c) = ab + ac$
- Additive and Multiplicative Identity $\underbrace{a + 0 = a}_{\text{defines } 0},\ \underbrace{a \cdot 1 = a}_{\text{defines } 1}$
- Additive and Multiplicative Inverses $\underbrace{a + -a = 0}_{\text{integers}}\ \underbrace{a \cdot \frac{1}{a} = 1}_{\text{fractions}}$

<u>New!</u>
- Closure: $a + b,\ a - b,\ a \times b,\ a \div b$ are all rational numbers

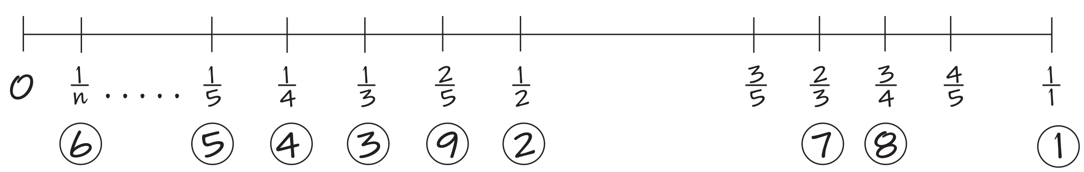
Note:
- Complete list of Arithmetic properties – Every statement, identity, rule, etc. follows from them.
- Used closure property throughout the course, but didn't make it explicit.

<u>Density</u>: How many fractions in the interval [0, 1]?

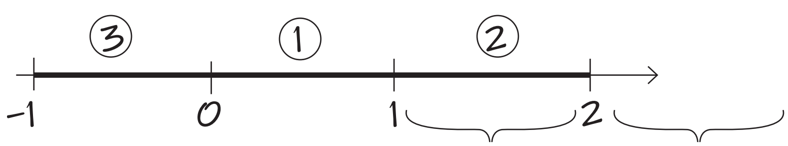
At least as many whole #'s: $0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots \frac{1}{n}$

More: $\frac{2}{3}, \frac{3}{4}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \ldots$

Fill-in in order
①, ②, ③ . . .



We could continue filling in pts forever



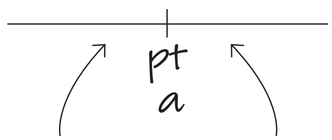
④ Can do the same for [–1, 0]
interval [1, 2], [2, 3], etc . . .

Leads to 2 questions:

(Q1) Is every pt on the number line a rational #?

[Answer Later]

(Q2) Given any pt on "   ", are there rational numbers "near by?"



Is there a rational # near a?

<u>Answer to Q 2</u>: Yes, because rational #'s are <u>dense</u>, i.e. any interval contains at least one rational #.

To see why,

Fix a point a and look at intervals

<u>rational in interval</u>?

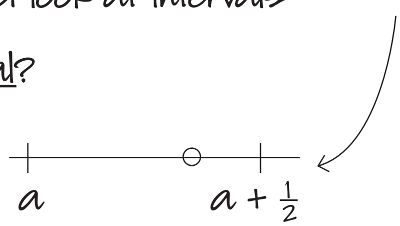
$[a, a + \frac{1}{2}]$

$[a, a + \frac{1}{3}]$

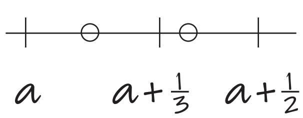
$[a, a + \frac{1}{4}]$

$[a, a + \frac{1}{n}]$

density $\Longrightarrow$ yes!

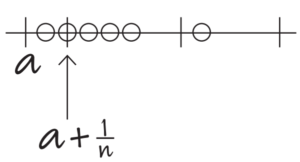


density $\Longrightarrow$ yes!



In fact, any interval contains an ∞ # of rationals!

density $\Longrightarrow$ yes!



How to find a rational in an interval:

<u>Ex</u> Find a rational # between 3.141 and π

$3.141\textcircled{0}0\ldots$

$3.141\textcircled{5}9\ldots$

<u>Pick any:</u>

3.1411

3.1412 ✓

3.1413

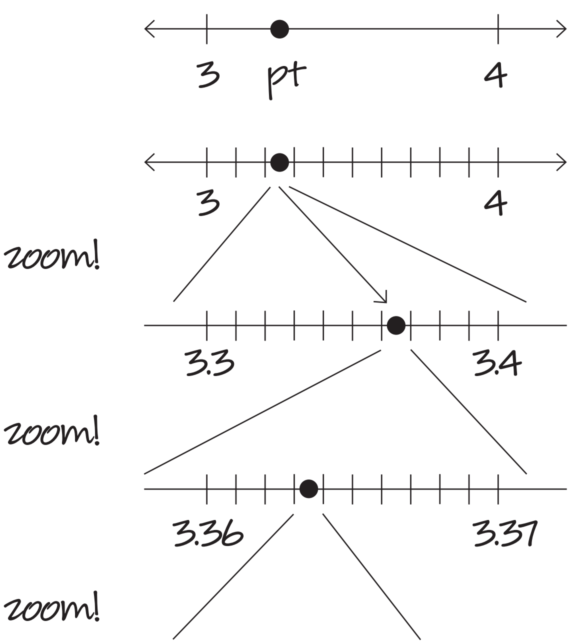
3.1414

$3.1412 = \frac{31412}{10000}$ ⟵ rational!

<u>Real numbers</u>

To Answer Q 1, we need to see that every pt on the #-line corresponds to a number.

<u>Algorithm for converting pts – to a number</u>



What # does this pt represent?

$3.3 \leq pt \leq 3.4$

$3.36 \leq pt \leq 3.37$

$3.364 \leq pt \leq 3.365$

Do this forever!

$pt = 3.364359682\ldots$

Called an <u>infinite decimal expansion</u>.

Note: Real numbers also satisfy the complete list of Arith. Prop.

<u>Ex</u> (a) $13 = 13.000\ldots$

$-.15 = -.15000\ldots$

$\frac{1}{3} = .3333\ldots$

$.\overline{183} = .183183183\ldots$

<u>Correspondence Thm</u>

repeating decimals $\Longleftrightarrow$ rationals! (± fractions)

Every rational number is also a real number.

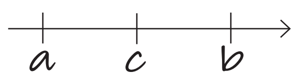
(b) $.101001000100001\ldots$ No repetend!

By correspondence Thm, this is <u>not</u> rational. Called an <u>irrational number</u> because it can't be written as $\frac{a}{b}$ for any integers a, b.

Answer to Q 1 : NO!

<u>Thm</u> Irrational number are also dense.

<u>Proof</u>: Given an interval [a,b] find a finite decimal c in the interval.



$\underbrace{.137\ldots6}_{\text{decimal \#c}}\ \underbrace{0000\ldots0}_{\text{enough 0's to stay in interval}}\ \underbrace{12112111211112}_{\text{any non repeating sequence.}}\ldots$ □

<u>Ex</u> [HW Prob] Find it irrational between $.\overline{67}$ and $.\overline{68}$

$.67676767\ldots$

$.68686868\ldots$

$.\overline{67} < .6\underline{80} < .\overline{68}$

↑ rational

Irrational: $.680000123456789101112\ldots$

Conclusions:

- There are 2 types of real numbers:
  rational and irrational.
- Both rationals and irrationals are dense
  (infinitely many in any interval).

| HW | Read § 9.3 do HW set 39