What to bring to class:
Ask students to bring PM 4A and 5A.

9.2 Fractions and Decimals [$1\frac{1}{2}$ days]

I. Converting Decimals to Fractions.

Denominator = appropriate power of 10

$.37 = \frac{37}{100}$

$3.288 = \frac{3288 \div 8}{1000 \div 8} = \frac{411}{125}$

$1000 = 2^3 \cdot 5^3$

Denominator = Product of 2's and 5's

Numerator = When no 2's or 5's, fraction in simplest form.

II. Fraction ⟶ decimal.

(a) Use equivalent fractions until denominator is a power of 10.

$\frac{71}{100} = .71$

$\frac{13}{25} = \frac{52}{100} = .52$

Student does ⟶ $\frac{3}{8} = \frac{3 \cdot 5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{375}{1000} = .375$

⟶ $\frac{7}{20} =$

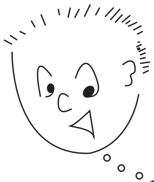
or (b) Just divide

$\frac{1}{8} = 8\overline{)1.00}$ = .125

```
     .125
 8 | 1.00
      8
     .20
      16
       40
```

$\frac{1}{3} = .3333\ldots\ldots$

Two types of decimal numbers:



.382 terminating (ends)

.333 . . . nonterminating (no end)

Must always check!

<u>Thm</u> A fraction $\frac{a}{b}$ in simplest form has a terminating decimal expansion $\Longleftrightarrow$ denominator b is a product of 2's and 5's only.

<u>Ex</u> $\frac{2}{25}$ $\frac{6}{150}$ $\frac{5}{321}$ $\frac{21}{35}$

<u>Sketch of Proof</u>: One way " $\Longrightarrow$ " A fraction with a terminating decimal expansion can be written $\frac{a}{b} = \frac{\text{whole \#}}{\text{power of 10}} = \frac{\text{whole \#}}{10^n} = \frac{\text{whole \#}}{2^n \cdot 5^n}$

<u>Ex</u> $.882 = \frac{882}{10^3} = \frac{882}{2^3 \cdot 5^3} = \frac{441}{2^2 \cdot 5^3}$

Power of 10

There may be cancelation, but denominator still product of 2's, 5's.

<u>Other way</u>: If fraction has form say $\frac{N}{2^3 \cdot 5^7}$ multiply by $\frac{2}{2}$ or $\frac{5}{5}$ until powers of 2 and 5 match in denominator.

$$\frac{N}{2^3 \cdot 5^7} = \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{2}{2} \cdot \frac{N}{2^3 \cdot 5^7} = \frac{2^4 \cdot N}{2^4 \cdot 2^3 \cdot 5^7} = \frac{16N}{2^7 \cdot 5^7}$$

$= \frac{16N}{10^7} \longleftarrow$ terminating decimal

If denominator has factors other than 2's or 5's, what happens?

<u>Ex</u>

$\frac{8}{37} = .216\ 216\ 216\ldots$

$= .\overline{216}$

↑ Period 3

Repeating digits called the <u>repetend</u>.

```
       .216216216
  37 ) 8.000
      -74
        60      ← pattern repeats forever
       -37
       230
      -222
        80      (repeats)
       -74
        60
       -37
       230
      -222
         8      (repeats)
```

<u>Ex</u> [Students do]

$\frac{9}{22} = .4\overline{09}$

↑ period 2

```
       .409090
  22 ) 9.0
       88
        200   (repeat)
        198
         200  (repeat)
        -198
          20
```

<u>Ex</u> $\frac{1}{7} = .\overline{142857}$

Split amongst class:

$\frac{2}{7} = .\overline{285714}$

$\frac{3}{7} =$

$\frac{4}{7} =$

$\frac{5}{7} =$

$\frac{6}{7} =$

Ask: The period in each case is <u><</u> <u>?</u>

```
       .142857
   7 ) 1.0
       7
       30
      -28
        20
        14
         60
        -56
          40
         -35
           50
          -49
            1   (repeat)
```

Basic Fact: Every fraction can be represented as a repeating decimal.

Proof: To convert $\frac{a}{b}$ to a decimal, we find $a \div b$ or $b\overline{)a}$. At each step in the long division the remainder r is a whole # with $0 \leq r <$ denominator.

[Explain by circling remainders in $\frac{1}{7}$ case.]

- $\Longrightarrow$ only b possibilities for r
- $\Longrightarrow$ repeats after at most b steps.
- $\Longrightarrow$ once a remainder repeats, long division steps must repeat. □

2 Cases:

- If some remainder r = 0, then it is a terminating decimal (still regard as repeating $\frac{1}{2} = .5 = .50000\ldots = .5\overline{0}$)
- Otherwise repeats with period $\leq b - 1$

Ex [If time, do calc. If not, write answer]

$\frac{1}{17}$

[Make table before you start!]

17	1
34	2
51	3
68	4
85	5
102	6
119	7
136	8
153	9

```
      .0588235294117647
17 ) 1.000000
    -85
     150
    -136
      140
     -136
        40
       -34
         60
        -51
          90
         -85
           50
          -34
            160
           -153
              70
             -68
```

```
20
-17
 30
 -17
  130
 -119
   110
  -102
     80
    -68
     120
    -119
       1
```

One can't see $\frac{1}{17}$ repeats on a calculator! This important concept can only be understood by students who know long division.

<u>Ex</u> $\frac{1}{13} = .\overline{076923}$

period is 6 (not 12)

<u>Other way:</u> Repeating decimals $\longrightarrow$ fractions!

<u>Ex</u> Write $.\overline{17}$ as a fraction

set $x = .\overline{17}$ ← constant

$$
\begin{aligned}
100x &= 17.171717\ldots \\
-x &= -.171717\ldots \\
\hline
99x &= 17 \\
x &= \tfrac{17}{99} \Longrightarrow \boxed{.\overline{17} = \tfrac{17}{99}}
\end{aligned}
$$

<u>Ex</u> $.\overline{361}$ ← students try $\qquad x = .\overline{361}$

$$
\begin{aligned}
1000x &= 361.361361\ldots \\
-x &= -.361361\ldots \\
\hline
999x &= 361 \longleftarrow \text{whole \#!} \\
x &= \tfrac{361}{999} \Longrightarrow \boxed{.\overline{361} = \tfrac{361}{999}}
\end{aligned}
$$

[Caution: "rule" $\frac{a}{999}$ only true if there are no initial non repeating digits.]

<u>Ex</u> $.11\overline{32} = \frac{1121}{9900}$

$$
\begin{aligned}
10000x &= 1132.3232\ldots \\
-100x &= -11.3232\ldots \\
\hline
9900x &= 1121 \\
x &= \tfrac{1121}{9900}
\end{aligned}
$$

Ex

$.2\overline{31}$

$5.\overline{173}$

$1.2\overline{34}$

$.9$

} Students do (especially) → .9

Fraction – Decimal Theorem

(a) Every fraction can be written as a repeating decimal and vice versa.

(b) The decimal form terminates $\Longleftrightarrow$ in simplest form the denominator is a product of 2's and 5's only.

$\longrightarrow$ Otherwise repeats of period $\leq$ (denominator -1)

Note: Terminating decimals have 2 repeating forms

$$1 = \begin{cases} 1.0000\ldots \\ .9999\ldots \end{cases} \qquad \frac{1}{4} = .25 = \begin{cases} .250000\ldots \\ .249999\ldots \end{cases}$$

[Fraction $\longleftrightarrow$ decimal correspondence has no other ambiguity].

HW Read § 9.2 do HW set 37